المادة الهياكل المتقطعة/ المرحلة الاولى

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SETS AND ELEMENTS

A set is a collection of objects called the elements or members of the set. The ordering of the elements is not important and repetition of elements is ignored, for example {1, 3, 1, 2, 2, 1} = {1, 2, 3}.
One usually uses capital letters, A,B,X, Y, . . . , to denote sets, and lowercase letters, a, b, x, y, . . ., to denote elements of sets.
Below you'll see just a sampling of items that could be considered as sets:

- The items in a store
- The English alphabet
- Even numbers

A set could have as many entries as you would like. It could have one entry, 10 entries, 15 entries, infinite number of entries, or even have no entries at all! For example, in the above list the English alphabet would have 26 entries, while the set of even numbers would have an infinite number of entries.
Each entry in a set is known as an **element or member**
Sets are written using curly brackets "{" and "}", with their elements listed in between. For example the English alphabet could be written as
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z}
2
and even numbers could be {0,2,4,6,8,10,...} (Note: the dots at the end indicating that the set goes on infinitely)

Principles:

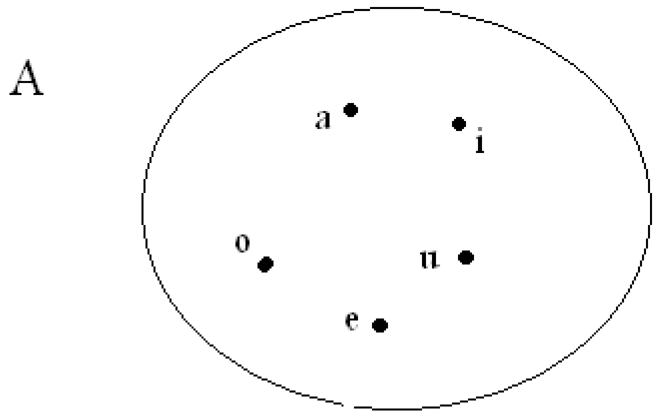
$\in$ belong to

$\notin$ not belong to

$\subseteq$ subset

$\subset$ proper subset, For example, {a, b} is a proper subset of {a, b, c}, but {a, b, c} is not a proper subset of {a, b, c}.

$\not\subset$ not subset
So we could replace the statement "a is belong to the alphabet" with a $\in$ {alphabet} and replace the statement "3 is not belong to the set of even numbers" with 3 $\notin$ {Even numbers}
Now if we named our sets we could go even further. Give the set consisting of the **alphabet** the name A, and give the set consisting of **even numbers** the name E. We could now write a $\in$ A and 3 $\notin$ E. **Problem** Let A = {2, 3, 4, 5} and C = {1, 2, 3, . . ., 8, 9}, Show that A is a proper subset of C. **Answer** Each element of A belongs to C so A $\subseteq$ C. On the other hand, 1 $\in$ C but 1 $\notin$ A. Hence A $\neq$ C. Therefore A is a proper subset of C.
There are three ways to specify a particular set:
1) By list its members separated by commas and contained in braces{ }, (if it is possible), for example, A= {a,e,i,o,u}
2) By state those properties which characterize the elements in the set, for example, A={x:x is a letter in the English alphabet, x is a vowel}
3) Venn diagram: (A graphical representation of sets).



Example (1)
A={x:x is a letter in the English alphabet, x is a vowel} e $\in$ A (e is belong to A) f $\notin$ A (f is not belong to A)
Example (2)
X is the set {1,3,5,7,9} 3 $\in$ X and 4 $\notin$ X
Example (3)
Let E = {x | $x^2 - 3x + 2 = 0$} → (x-2)(x-1)=0 → x=2 & x=1
E = {2, 1}, and 2$\in$E

Universal set, empty set:

In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the universal set. For example, in human population studies the universal set consists of all the people in the world. We will let the symbol U denotes the universal set.
The set with no elements is called the empty set or null set and is denoted by $\varnothing$ or {}

Subsets:

Every element in a set A is also an element of a set B, then A is called a subset of B. We also say that B contains A. This relationship is written:

$A \subset B$ or $B \supset A$

If A is not a subset of B, i.e. if at least one element of A dose not belong to B, we write $A \not\subset B$.

Example 4:

Consider the sets. A = {1,3,4,5,8,9} B = {1,2,3,5,7} and C ={1,5}

Then $C \subset A$ and $C \subset B$ since 1 and 5, the element of C, are also members of A and B. But $B \not\subset A$ since some of its elements, e.g. 2 and 7, do not belong to A. Furthermore, since the elements of A,B and C must also belong to the universal set U, we have that U must at least the set {1,2,3,4,5,7,8,9}.

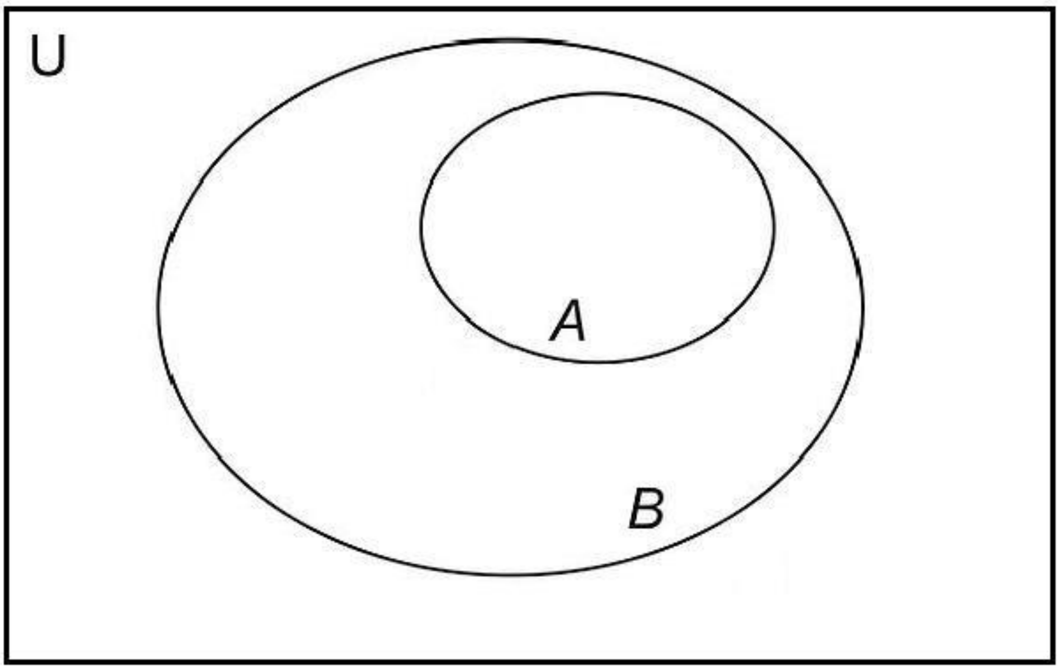
$A \subset B$: $\{\forall\ x \in A \Rightarrow x \in B$

$A \not\subset B$: $\{\exists\ x \in A$ but $x \notin B$

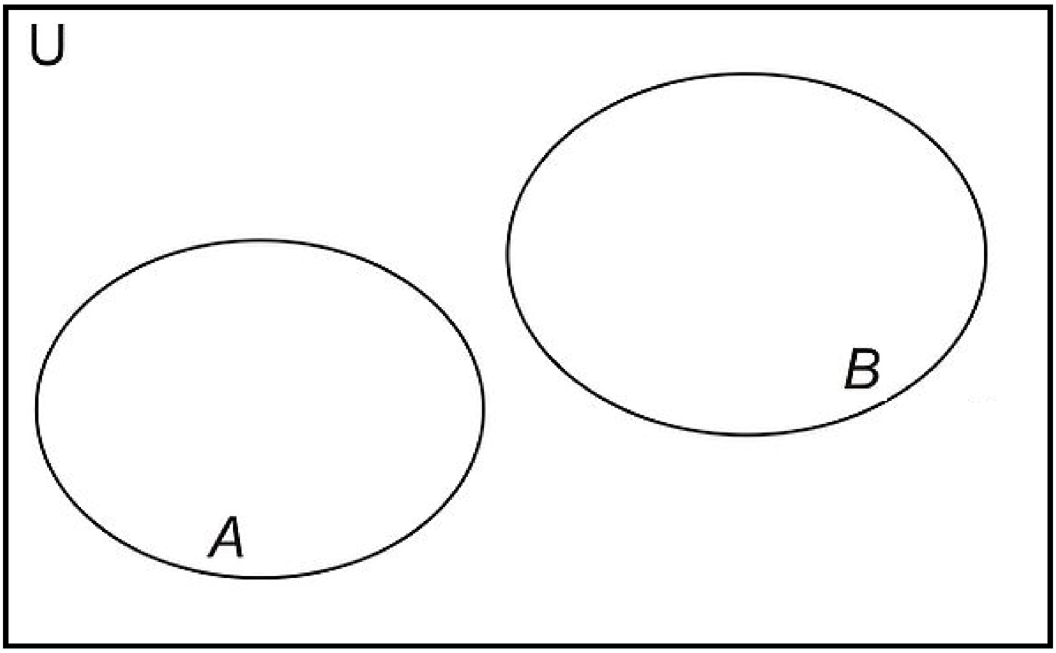
$\forall$: For all لكل

$\exists$: There exists الاقل على يوجد

The notion of subsets is graphically illustrated below:



A is entirely within B so $A \subset B$.



A and *B* are disjoint or ($A \cap B = \emptyset$) so we could write $A \not\subset B$ and $B \not\subset A$.

Set of numbers:

Several sets are used so often, they are given special symbols.

N = the set of *natural numbers* or positive integers

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Z = the set of all integers: . . . ,−2,−1, 0, 1, 2, . . .

$$\mathbb{Z} = \mathbb{N} \cup \{\ldots, -2, -1\}$$

Q = the set of rational numbers

$$\mathbb{Q} = \mathbb{Z} \cup \{\ldots, -1/3, -1/2, 1/2, 1/3, \ldots, 2/3, 2/5, \ldots\}$$

Where $Q = \{ a/b : a , b \in Z, b \neq 0\}$

R = the set of real numbers

$$\mathbb{R} = \mathbb{Q} \cup \{\ldots, -\pi, -\sqrt{2}, \sqrt{2}, \pi, \ldots\}$$

C = the set of complex numbers

$$\mathbb{C} = \mathbb{R} \cup \{i, 1+i, 1-i, \sqrt{2}+\pi i, \ldots\}$$

Where $C=\{ x + iy ; x , y \in R; i = \sqrt{-1}\}$

Observe that $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$.

Theorem 1:

For any set A, B, C:

1- $\varnothing \subset A \subset U$.
2- $A \subset A$.
3- If $A \subset B$ and $B \subset C$, then $A \subset C$.
4- $A = B$ if and only if $A \subset B$ and $B \subset A$.

Set operations:

1) UNION:

The *union* of two sets *A* and *B*, denoted by $A \cup B$, is the set of all elements which

belong to *A* or to *B*;

$A \cup B = \{ x : x \in A \text{ or } x \in B\}$

Example

A={1,2,3,4,5} B={5,7,9,11,13}

$A \cup B = \{1,2,3,4,5,7,9,11,13\}$

2) **INTERSECTION**

The *intersection* of two sets *A* and *B*, denoted by $A \cap B$, is the set of elements which belong to both *A* and *B*;

$A \cap B = \{ x : x \in A \text{ and } x \in B\}$.

Example 1

A={1,3,5,7,9} B={2,3,4,5,6}

The elements they have in common are 3 and 5

$A \cap B = \{3,5\}$

Example 2

A={The English alphabet} B={vowels}

So $A \cap B$ = {vowels}

Example 3

A={1,2,3,4,5} B={6,7,8,9,10}

In this case A and B have nothing in common. $A \cap B = \varnothing$

3) THE DIFFERENCE:

The difference of two sets A\B or A-B is those elements which belong to A but which do not belong to B.

$A \backslash B = \{x : x \in A, x \notin B\}$

4) COMPLEMENT OF SET: Complement of set A^c or A' , is the set of elements which belong to U but which do not belong to A .

$A^c = \{x : x \in U, x \notin A\}$

Example: let A={1,2,3}

B={3,4}

U={1,2,3,4,5,6}

Find:

$A \cup B = \{1, 2, 3, 4\}$

$A \cap B = \{3\}$

$A - B = \{1, 2\}$

$A^c = \{4, 5, 6\}$

5) Symmetric difference of sets

The symmetric difference of sets A and B denoted by A B, consists of those elements which belong to A or B but not to both. That is,

A B = $(A \cup B)\backslash(A \cap B)$ or A B = $(A\backslash B) \cup (B\backslash A)$

Example: Suppose U = N = {1, 2, 3, . . .} is the universal set.
Let A = {1, 2, 3, 4}, B= {3, 4, 5, 6, 7}, C= {2, 3, 8, 9}, E= {2, 4, 6, 8,. . .}
Then:
A^c = {5, 6, 7, . . .}, B^c = {1, 2, 8, 9, 10, . . .}, C^c ={1,4,5,6,7,10,…} E^c ={1, 3, 5, 7, ...}
A\B = {1, 2}, A\C = {1, 4}, B\C = {4, 5, 6, 7}, A\E = {1, 3},
B\A = {5, 6, 7}, C\A = {8, 9}, C\B = {2, 8, 9}, E\A = {6, 8, 10, 12, . . .}.
Furthermore:
A B = (A\B) ∪ (B\A) = {1, 2, 5, 6, 7}, B C = {2, 4, 5, 6, 7, 8, 9},
A C = (A\C) ∪ (B\C) = {1, 4, 8, 9}, A E = {1, 3, 6, 8, 10, . . .}.

Theorem 2 :
$A \subset B$, $A \cap B = A$, $A \cup B = B$ are equivalent
Theorem 3: (Algebra of sets)
Sets under the above operations satisfy various laws or identities which are listed below:
1- $A \cup A = A$
$A \cap A = A$
2- $(A \cup B) \cup C = A \cup (B \cup C)$ Associative laws
$(A \cap B) \cap C = A \cap (B \cap C)$
3- $A \cup B = B \cup A$ Commutativity
$A \cap B = B \cap A$
4- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Distributive laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5- $A \cup \varnothing = A$ Identity laws
$A \cap U = A$
6- $A \cup U = U$ Identity laws
$A \cap \varnothing = \varnothing$
7- $(A^c)^c = A$ Double complements
8- $A \cup A^c = U$ Complement intersections
and unions
$A \cap A^c = \varnothing$
9- $U^c = \varnothing$
$\varnothing^c = U$
10- $(A \cup B)^c = A^c \cap B^c$ De Morgan's laws
$(A \cap B)^c = A^c \cup B^c$
We discuss two methods of proving equations involving set operations. The first is to break down what it means for an object x to be an element of each side, and the second is to use Venn diagrams.
For example, consider the first of De Morgan's laws:
$(A \cup B)^c = A^c \cap B^c$
We must prove: 1) $(A \cup B)^c \subset A^c \cap B^c$
2) $A^c \cap B^c \subset (A \cup B)^c$
We first show that $(A \cup B)^c \subset A^c \cap B^c$

Let's pick an element at random $x \in (A \cup B)^c$. We don't know anything about x, it could be a number, a function. All we do know about x, is that:
$x \in (A \cup B)^c$, so

$x \notin A \cup B$
because that's what complement means. Therefore
$x \notin A$ and $x \notin B$,
by pulling apart the union. Applying complements again we get
$x \in A^c$ and $x \in B^c$
Finally, if something is in 2 sets, it must be in their intersection, so
$x \in A^c \cap B^c$
So, any element we pick at random from: $(A \cup B)^c$ is definitely in, $A^c \cap B^c$, so by definition
$(A \cup B)^c \subset A^c \cap B^c$
Next we show that $(A^c \cap B^c) \subset (A \cup B)^c$.
This follows a very similar way. Firstly, we pick an element at random from the first set, $x \in (A^c \cap B^c)$
Using what we know about intersections, that means
$x \in A^c$ and $x \in B^c$
Now, using what we know about complements,
$x \notin A$ and $x \notin B$.
If something is in neither A nor B, it can't be in their union, so
$x \notin A \cup B$,
And finally
$\therefore\ x \in (A \cup B)^c$
We have prove that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$ and that every element of $A^c \cap B^c$ belongs to $A \cup B)^c$. Together, these inclusions prove that the sets have the same elements, i.e. that $(A \cup B)^c = A^c \cap B^c$

Power set

The power set of some set S, denoted P(S), is the set of all subsets of S (including S itself and the empty set)
Example 1: Let A = { 1,2 3}
Power set of set A = P(A) = [{1},{2},{3},{1,2},{1,3},{2,3},{},A]
Example 2: P({0,1})={{},{0},{1},{0,1}}

Classes of sets: Collection of subset of a set with some properties

Example: Suppose A = { 1,2 3} , let X be the class of subsets of A which contain exactly two elements of A. Then
class X = [{1,2},{1,3},{2,3}]

Cardinality

The cardinality of a set S, denoted **|S|**, is simply the number of elements a set has. So |{a,b,c,d**}| =** 4, and so on. The cardinality of a set need not be finite: some sets have infinite cardinality.

The cardinality of the power set

Theorem: If |A| = n then |P(A)| = 2^n (Every set with n elements has 2^n subsets)

Problem set write the answers to the following questions. 1. |{1,2,3,4,5,6,7,8,9,0}|
2. |P({1,2,3})|

3. P({0,1,2})

4. P({1})

Answers

1. 10

2. 2^3=8

3. {{},{0},{1},{2},{0,1},{0,1,2},{0,2},{1,2}}

4. {{},{1}}

The Cartesian product

The Cartesian Product of two sets is the set of all tuples made from elements of two sets. We write the Cartesian Product of two sets A and B as A × B. It is defined as:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

It may be clearer to understand from examples;

$$\{0,1\} \times \{2,3\} = \{(0,2),(0,3),(1,2),(1,3)\}$$
$$\{a,b\} \times \{c,d\} = \{(a,c),(a,d),(b,c),(b,d)\}$$
$$\{0,1,2\} \times \{4,6\} = \{(0,4),(0,6),(1,4),(1,6),(2,4),(2,6)\}$$

Example: If A = {1, 2, 3} and B = {x, y} then
A . B = {(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)}
B . A = {(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)}

It is clear that, the cardinality of the Cartesian product of two sets A and B is:

$$|A \times B| = |A||B|$$

A Cartesian Product of two sets A and B can be produced by making tuples of each element of A with each element of B; this can be visualized as a grid (which *Cartesian* implies) or table: if, *e.g.*, A = { 0, 1 } and B = { 2, 3 }, the grid is

Problem set Answer the following questions:
1. {2,3,4}×{1,3,4}
2. {0,1}×{0,1}
3. |{1,2,3}×{0}|
4. |{1,1}×{2,3,4}|

Answers 1. {(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4)}
2. {(0,0),(0,1),(1,0),(1,1)}
3. 3
4. 6

Partitions of set:

Let S be any nonempty set. A partition (∏) of S is a subdivision of S into nonoverlapping, nonempty subsets. A partition of S is a collection {Ai} of non-empty subsets of S such that:
1) Ai ≠∅, where i=1,2,3,……
2) The sets of {Ai } are mutually disjoint
or Ai ∩ Aj = ∅ where i ≠ j.
3) UA_i = S, where $A_1 \cup A_2 \cup .... \cup A_i = S$

The partition of a set into five cells, A1, A2,A3,A4,A5, can be represented by Venn diagram

Example 1: let A = {1,2,3,n}
A1 = {1}, A2 = {3,n}, A3 = {2}
Π = {A1, A2, A3} is a partition on A because it satisfy the three above conditions.
Example 2 : Consider the following collections of subsets of S = {1,2,3,4,5,6,7,8,9}
(i) [{1,3,5},{2,6},{4,8,9}]
(ii) [{1,3,5},{2,4,6,8},{5,7,9}]
(iii) [{1,3,5},{2,4,6,8},{7,9}]
Then
(i) is not a partition of S since 7 in S does not belong to any of the subsets.
(ii) is not a partition of S since {1,3,5} and {5,7,9} are not disjoint.
(iii) is a partition of S.

FINITE SETS, COUNTING PRINCPLE:

A set is said to be finite if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise, a set is said to be infinite. For example, the empty set ∅ and the set of letters of English alphabet are finite sets, whereas the set of even positive integers, {2,4,6,…..}, is infinite.
If a set A is finite, we let *n*(A) or #(A) denote the number of elements of A.
Example: If A ={1,2,a,w} then
n(A) = #(A) = |A| = 4
Lemma: If A and B are finite sets and disjoint Then $A \cup B$ i

$$n(A \cup B) = n(A) + n(B)$$

Theorem (Inclusion–Exclusion Principle): Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and
13
$$|A \cup B| = |A| + |B| - |A \cap B|$$
That is, we find the number of elements in A or B (or both) by first adding n(A) and n(B) (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.
We can apply this result to obtain a similar formula for three sets:

Corollary:

If A,B,C are finite sets then
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example (1) :
A= {1,2,3}
B= {3,4}
C= {5,6}
$A \cup B \cup C = \{1,2,3,4,5,6\}$
$|A \cup B \cup C| = 6$
|A| =3 , |B| = 2 , |C| = 2
$A \cap B = \{3\}$, $|A \cap B| = 1$
$A \cap C = \{\ \}$, $|A \cap C| = 0$
$B \cap C = \{\ \}$, $|B \cap C| = 0$
$A \cap B \cap C = \{\ \}$, $|A \cap B \cap C| = 0$
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$|A \cup B \cup C| = 3 + 2 + 2 - 1 – 0 – 0 + 0 = 6$

Example (2):

Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students:

(a) only on list A

(b) only on list B

(c) on list $A \cup B$

Solution:

(a) List A has 30 names and 20 are on list B; hence $30 - 20 = 10$ names are only on list A.

(b) Similarly, $35 - 20 = 15$ are only on list B.

(c) We seek $n(A \cup B)$. By inclusion–exclusion,

$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45$.

Example (3):

Suppose that 100 of 120 computer science students at a college take at least one of languages: French, German, and Russian and:

65 study French (F).

45 study German (G).

42 study Russian (R).

20 study French & German $F \cap G$.

25 study French & Russian $F \cap R$.

15 study German & Russian $G \cap R$.

Find the number of students who study:

1) All three languages ($F \cap G \cap R$)

2) The number of students in each of the eight regions of the Venn diagram

Solution:

$|F \cup G \cup R| = |F| + |G| + |R| - |F \cap G| - |F \cap R| - |G \cap R| + |F \cap G \cap R|$

$100 = 65 + 45 + 42 - 20 - 25 - 15 + |F \cap G \cap R|$

$100 = 92 + |F \cap G \cap R|$

$\therefore |F \cap G \cap R| = 8$ students study the 3 languages

$20 - 8 = 12$ $(F \cap G) - R$

$25 - 8 = 17$ $(F \cap R) - G$

$15 - 8 = 7$ $(G \cap R) - F$

65 – 12 – 8 – 17 = 28 students study French only

45 – 12 – 8 7 = 18 students study German only

42 – 17 – 8 7 = 10 students study Russian only

120 – 100 = 20 students do not study any language